

On a Discrete Korovkin Theorem

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In [G. A. Anastassiou, A discrete Korovkin theorem, *J. Approx. Theory* 45 (1985), pp. 383-388, Theorem 3], a discrete Korovkin theorem was given. We restate the theorem here and its proof, correcting a mistake in the above reference. © 1990 Academic Press, Inc.

RESULTS

Our main result is the following

THEOREM 1. *Let $X = \{x_1, \dots, x_j, \dots\}$ be a set. Consider $B(X)$, the space of real valued bounded functions on X with the supremum norm $\|\cdot\|_\infty$, and a sequence of positive linear operators $L_n: B(X) \rightarrow B(X)$ such that $L_n(1, x_j) = 1$ for all j . Suppose that, for some $f_1, \dots, f_k \in B(X)$,*

$$\lim_{n \rightarrow \infty} L_n(f_i, x_j) = f_i(x_j) \quad \text{for all } i \text{ and } j. \quad (1.1)$$

In order that $L_n(f, x_j) \rightarrow f(x_j)$ for all $f \in B(X)$ and all j , it is enough to assume that for each j there are real constants β_1, \dots, β_k such that

$$\sum_{i=1}^k \beta_i (f_i(x) - f_i(x_j)) \geq 1 \quad \text{for all } x \in X - \{x_j\}. \quad (1.2)$$

We need the following

LEMMA 2. *Let $X = \{x_1, \dots, x_j, \dots\}$ be a measurable space, $1 \leq p < \infty$, and let μ be a finite positive measure on X such that $\mu(\{x_j\}) > 0$ for all j . Let $B(X)$ be as above, and let $f, f_1, f_2, \dots \in B(X)$, where all $\|f_n\|_\infty < c$, $0 < c < \infty$. Then $f_n \rightarrow f$ pointwise on X iff $f_n \rightarrow f$ in $L_p(X, \mu)$.*

Proof of Lemma 2. (\Rightarrow) By the uniform boundedness of f_n , as $\mu(X) < \infty$, we obtain $|f_n|^p \leq c^p \in L_1(X, \mu)$. Since $f_n \rightarrow f$ pointwise, by a variation of the dominated convergence theorem (see [3, p. 180]), we get $f_n \rightarrow f$ in $L_p(X, \mu)$. Note $B(X) \subseteq L_p(X, \mu)$.

(\Leftarrow) The L_p convergence implies weak convergence, the indicator function $I_{\{x_j\}} \in L_q(X, \mu)$ where $(1/p) + (1/q) = 1$, and $\mu(\{x_j\}) > 0$. Hence the pointwise convergence.

Next is an independent L_p result which will be used in the proof of Theorem 1.

PROPOSITION 3. Let $X = \{x_1, \dots, x_j, \dots\}$ be a set. Let $w(x_j) > 0$ for all j and $\sum_{j=1}^{\infty} w(x_j) < \infty$. Let $B(X)$ be as above, and let L_n be a sequence of positive linear operators: $B(X) \rightarrow B(X)$ such that $L_n(1, x_j) = 1$ for all j . Suppose that, for some $f_1, f_2, \dots, f_k \in B(X)$ and some $p, 1 \leq p < \infty$,

$$\lim_{n \rightarrow \infty} \left(\sum_{j=1}^{\infty} |L_n(f_i, x_j) - f_i(x_j)|^p \cdot w(x_j) \right) = 0, \quad i = 1, 2, \dots, k. \quad (3.1)$$

In order that $\sum_{j=1}^{\infty} |L_n(f, x_j) - f(x_j)|^p \cdot w(x_j) \rightarrow 0$ for all $f \in B(X)$, it is enough to assume the following: for each j there are real constants β_1, \dots, β_k such that

$$\sum_{i=1}^k \beta_i (f_i(x) - f_i(x_j)) \geq 1 \quad \text{for all } x \in X - \{x_j\}. \quad (3.2)$$

Proof of Proposition 3. The weight w gives rise to a positive finite measure μ on X with $\mu(\{x_j\}) > 0$ for all $x \in X$. Since $B(X) \subseteq L_p(X, \mu)$, (3.1) implies $\|L_n(f_i) - f_i\|_p \rightarrow 0$ for all i . If there exists $f \in B(X)$ such that $\|L_n(f) - f\|_p \not\rightarrow 0$, then there is j and a positive ε so that

$$|L_n(f, x_j) - f(x_j)| > \varepsilon \quad \text{for all } n \geq \text{some } n_0.$$

Because each positive linear functional $L_n(\cdot, x_j)$ on $B(X)$ is bounded, by a basic representation theorem, for each specific $j = j_0$ as above, there exists $g_{j_0, n} \in L_q(X, \mu)$, where $(1/p) + (1/q) = 1$, such that

$$L_n(f, x_{j_0}) = \int_X f(x) \cdot g_{j_0, n}(x) \cdot \mu(dx) \quad \text{for all } f \in B(X).$$

As $L_n(1, x_{j_0}) = 1$, the positivity of $L_n(\cdot, x_{j_0})$ implies $\int_X g_{j_0, n}(x) \cdot \mu(dx) = 1$ and $g_{j_0, n}(x) \geq 0$ for all $x \in X$. Thus

$$\begin{aligned} \varepsilon &< |L_n(f, x_{j_0}) - f(x_{j_0})| \\ &= \left| \int_{X - \{x_{j_0}\}} (f(x) - f(x_{j_0})) \cdot g_{j_0, n}(x) \cdot \mu(dx) \right| \\ &\leq \|f - f(x_{j_0})\|_{\infty} \cdot \left(\int_{X - \{x_{j_0}\}} g_{j_0, n}(x) \cdot \mu(dx) \right), \end{aligned}$$

so

$$\int_{X - \{x_{j_0}\}} g_{j_0, n}(x) \cdot \mu(dx) > \frac{\varepsilon}{\|f - f(x_{j_0})\|_\infty} =: \delta > 0 \quad \text{for all } n \geq n_0.$$

There cannot be real constants $\beta_1, \beta_2, \dots, \beta_k$ with

$$\sum_{i=1}^k \beta_i \cdot (f_i(x) - f_i(x_{j_0})) \geq 1 \quad \text{for all } x \in X - \{x_{j_0}\}.$$

Since, otherwise, we would have

$$\sum_{i=1}^k \beta_i \cdot (f_i(x) - f_i(x_{j_0})) \cdot g_{j_0, n}(x) \geq g_{j_0, n}(x),$$

for all $x \in X - \{x_{j_0}\}$, and therefore

$$\begin{aligned} \sum_{i=1}^k \beta_i \cdot \int_{X - \{x_{j_0}\}} (f_i(x) - f_i(x_{j_0})) \cdot g_{j_0, n}(x) \cdot \mu(dx) \\ \geq \int_{X - \{x_{j_0}\}} g_{j_0, n}(x) \cdot \mu(dx). \end{aligned}$$

(Note that $L_n(f_i, x_{j_0}) = \int_X f_i(x) \cdot g_{j_0, n}(x) \cdot \mu(dx)$, $i = 1, \dots, k$.) Consequently, since $L_n(f_i, x_{j_0}) \rightarrow f_i(x_{j_0})$, $i = 1, \dots, k$, we would get

$$0 = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^k \beta_i \cdot (L_n(f_i, x_{j_0}) - f_i(x_{j_0})) \right) > \delta > 0.$$

Proof of Theorem 1. Note that $B(X) \subseteq L_p(X, \mu)$ for every $p \in [1, \infty)$, and every finite measure μ on X for which each $\mu(\{x_j\}) > 0$. By Lemma 2, the pointwise convergence $L_n(f_i, x_j) \rightarrow f_i(x_j)$, $i = 1, \dots, k$, for all j , is equivalent, for such p and μ , to the convergence in $L_p(X, \mu)$ of $L_n(f_i)$ to f_i , $i = 1, \dots, k$. Furthermore, this measure μ can serve as a weight function on X . Thus Proposition 3 implies our theorem.

REFERENCES

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